Mathematics Teacher

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MATHEMATICS AND SUNSHINE [1]

BY H. E. SLAUGHT
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I. SUNSHINE

If the average man in the street were asked to name the benefits derived from sunshine, he would probably say "light and warmth" and there he would stop. But, if we analyze the matter a little more deeply, we will soon realize that sunshine is the one great source of all forms of life and activity on this old planet of ours. To mention only a few particulars, wood, peat, coke, soft coal, anthracite, kerosene, fuel oil, gas, gasoline, are one and all forms of canned sunshine in varying degrees of transformation. Hence I cannot stoke my furnace, turn on my oil-burner, step on the gasoline in my automobile, cook my meals by wood, gas, oil or coal, light my house with kerosene or gas, without drawing upon the sunshine which was stored up for me in the form of carbon long ages ago in the deep recesses of the earth. All the industries of the world, great and small, all transportation by rail, water or air, are made possible only through the products of sunshine. Even the wind which drives my sailboat is only the rushing currents of sun-heated air.

But someone rises to say that *now* we can do all these things by electricity and so free ourselves from our great burden of debt to the sunshine. Not so fast—whatever the ultimate nature of electricity, we know only one way of producing it for commercial uses, namely, by means of dynamos driven by steam power or water power, and we could have neither of these sources of power except for the sunshine. What do you say! We could not have a waterfall without sunshine? Surely, there could be no fall of water unless the water had first been *uplifted* in the form of

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[1] Abstract of an address delivered at the meeting of the National Council of Teachers of Mathematics, Boston, February 25, 1928.

vapor by the sun's heat, condensed into clouds and deposited in the mountains in the form of the everlasting snows, thence to melt and feed the rivers in their downward course.

Even the graphite of my eversharp pencil and the diamond of my scarf pin are forms of that same carbon which gave us the coal away back in the carboniferous age. The very air which we breathe is life-giving only when it has been purified in the sunshine of the great out-of-doors. And not only do we get health and vigor *indirectly* from the sunshine through the air which we breathe, but these blessings come *directly* to us through the contact of the sun's rays with our bodies—the so-called actinic, or ultra-violet, rays which will not penetrate the glass in our windows but must be absorbed directly from old Sol himself in the open air of heaven.

Not only does sunshine furnish the power to do all the mechanical work of the world, not only does it supply the source of all life-giving energy to both plants and animals, but it also supplies the source of all beauty due to colors and the combinations of colors. These colors are all wrapped up in the rays of light as they come from the sun and are spread out for us in the varying wave-lengths which produce the color sensations on the retinas of our eyes.

II. Mathematics

It is the purpose of this paper to support the proposition that mathematics underlies present-day civilization in much the same far-reaching manner as sunshine underlies all forms of life, and that we unconsciously share the benefits conferred by the mathematical achievements of the race just as we unconsciously enjoy the blessings of the sunshine.

If you ask the average man in the street what are the benefits derived from mathematics, he would probably mention the "adding machine and the compound interest tables" and there he would stop. It is true that these two devices are the agents by which a very large part of the enormous business of the day is transacted, and it is also true that the computing machines and compound interest tables were made possible by some very elegant applications of mathematics which, however, one does not need to know in order to enjoy the benefits of their use, just as one does not need to know about the ultra-violet rays of the sun in order to enjoy the benefits of their healing and life-giving properties.

Probably not one in twenty of those who use an annuity table could produce offhand the mathematics on which it is based.

But the mathematician distinguishes very sharply between *computation* and the *mathematical principles* upon which the computation is based. The latter alone is what is properly called mathematics. For instance, a computer in the office of the Nautical Almanac at the Naval Observatory in Washington, where all the data of eclipses, tides, chronometer corrections, positions of the planets, comets, etc., are made and kept, need not himself be a great mathematician, but the fundamental principles on which all such computations depend have been developed and organized by some of the most profound mathematical research.

III. How Mathematics Underlies Civilization

In order to show the fundamental character of mathematics in present-day civilization, consider the following examples:

1. *Mathematics Reveals the Heavens to Us.*—To the ancients the motions of the heavenly bodies were mysterious and baffling. To them the earth was the center of the universe and the sun, planets, and stars revolved about it. It was mathematics in the hands of giant intellects like Sir Isaac Newton that gave us the true understanding of these matters and made the heavens an open book for our guidance. For example, the chronometers of the world are regulated by the stars and now the radio carries these time signals to the remotest parts of the earth. The ocean tides are governed by the motion and positions of the moon and the most abstruse mathematical calculations are needed to make the tide tables that are so necessary for the guidance of the mariner. In fact all navigation of the oceans and great lakes is absolutely dependent upon information and guidance derived from the heavens and based upon mathematical calculation.

2. *Mathematics Reveals the Earth to Us.*—How is it possible to make an accurate map of the jagged New England coast line or of the rugged Colorado mountain country, where an observer can see only a very small portion of the landscape at a time, and where direct measurement of distances, such as cape to cape, or mountain peak to mountain peak, cannot be made? These things are impossible without the help of mathematical principles and calculation, but with such help the accuracy of our coast survey and of our mountain and prairie mapping is almost beyond be-

lief. Without such maps our knowledge of the earth's surface
would be crude and ineffective. It is well known that the New
England coast survey involved thousands of mathematical com-
putations running back to a single *measured* ten-mile base line
up in Maine and extending down to a *computed* base line on Long
Island; and that a subsequent measurement of the latter showed
a difference of only a few inches in ten miles as compared with
the computed distance.

3. *Mathematics is a Chief Requisite in Our Army, Navy, and
Air Service.*—The greater part of the scientific training at West
Point and Annapolis rests upon mathematics, and the science of
aeronautics involves the most recondite mathematical investiga-
tions. In 1917 the boys who were suddenly thrust into prepara-
tion for war service soon discovered that mathematics was the
chief requisite for any commission in the army or navy; and in
Washington the war department soon found that it needed the
expert service of a score or more of the best mathematicians of
the country in order to put the Ordnance equipment in shape for
effective service. As a result of that emergency experience the
war and navy departments since that time have been sending
carefully selected young men to various university centers, espe-
cially to the University of Chicago, for advanced training in
mathematics. As a natural sequence to these events, the Univer-
sity of Chicago has recently published a treatise on "Exterior
Ballistics" by Dr. F. R. Moulton which marks a new era in the
application of higher mathematics to the behavior of projectiles.

4. *Mathematics is Fundamental to All the Physical Sciences.*—
It will be at once admitted that all developments of physics and
mechanics rest upon mathematics; for instance, stresses, strains
and strengths of materials; the construction of bridges, arches
and tunnels; the laws of falling bodies and of harmonic motion;
the expansion of gases; the principles of statics and dynamics,
etc. But it may not be so readily admitted that chemistry and
electricity are so closely connected with mathematics. However,
the research student in these fields soon realizes that mathematics
is absolutely essential to his progress. For instance, it is said
that the greatest achievement of Charles P. Steinmetz, formerly
chief engineer of the General Electric Company, was his develop-
ment of the mathematics of the alternating current, involving as
it did functions of a complex variable and other advanced topics.

Without such understanding of the mathematical basis of electricity, its present amazing development would have been impossible. It is well known that one of the major interests of the Rockefeller Foundation is the promotion of public health through scientific research in the fields underlying medicine. Their procedure is to delegate to the National Research Council the selection of highly trained men whose powers of research in these fields have already been tested and to award them cash fellowships as an incentive to still further prosecute their investigations. The biological sciences, of course, were chosen initially for these fellowship awards, then chemistry as underlying biology, and physics as underlying chemistry, and finally *mathematics* as underlying all the rest.

5. *Mathematics is an Important Tool in Many of the Biological Sciences.*—It is surprising even to mathematicians to learn from scientific workers in physiology, biometry, and biochemistry how much and how high mathematics they need in their investigations. This was brought out some years ago in connection with a symposium in New York City on the "Contributions of Mathematics to other Sciences." It was recently emphasized at Chicago when a prominent diagnostician and research worker in physiology, a man with a Harvard Ph.D. in this subject, as well as a medical degree, became a regular attendant upon the classes in calculus because, as he said, he found himself continually hampered in his research for lack of ability to handle mathematics, a subject which he had deliberately neglected as an undergraduate. Any subject which admits of a statistical treatment —and this is becoming more and more effective with the biological sciences as well as with economics and the social sciences— thereby classifies itself as dependent upon mathematics for its development. In fact the mathematics of statistics has become one of the most important subjects in the science curriculum. In general, the more highly organized any science becomes, the more mathematical is its structure and the more necessary is its dependence upon mathematics.

6. *Mathematics is an Important Element in Many Forms of Beauty.*—When we speak of music which is beauty of tone-harmony, of art which is beauty of form and color, of architecture which is beauty of symmetry and structure, we are actually using mathematical terms somewhat disguised. Harmony, form,

structure, symmetry—these are concepts of mathematical content and their full meaning can be understood only from the mathematical standpoint. Even mathematics itself is full of beauty when thought of in connection with these same concepts of form and symmetry. For example, the manifold relations among the trigonometric functions, the coordination of algebra and geometry in one of the most beautiful of elementary mathematical subjects—analytic geometry—or the wonderful properties of conic sections as developed in synthetic projective geometry, these and hundreds of other topics in mathematics are examples of real beauty, the beauty of form and symmetry, of consistency and coordination.

It would seem that enough has been said to show that mathematics pervades and affects the affairs of men in all their varied activities as certainly as the sunshine pervades the earth and supports the life of all animate things. The importance of mathematics is gradually seeping through to the public as is evidenced by the many research laboratories now financed by big commercial concerns and by the many appeals of smaller concerns for help in their efficiency programs. For instance, a Chicago box manufacturing company willingly paid a thousand dollars to some mathematicians for a small mathematical chart which they found would save the work of two or three assistants.

IV. IMPORTANCE OF THE WORK OF THE NATIONAL COUNCIL

It is our business as teachers to "sell mathematics to the public." To do this we need to believe in it most thoroughly ourselves, to proclaim it on every reasonable occasion, such as assembly talks, parent-teacher discussions, club meetings, etc., and to inspire our pupils with the propaganda that mathematics is the most marvelous and most powerful achievement of the human race. While this responsibility rests upon us as individuals, it is still more important that we should combine our efforts in such an organization as the National Council. The future of mathematics in America rests largely with three great organizations: the American Mathematical Society which is a purely research body, the Mathematical Association of America which is concerned primarily with the teaching of mathematics in the collegiate field, and the National Council of Teachers of Mathematics which is a federation of clubs and associations in the ele-

mentary and secondary fields, organized seven years ago to carry on the spirit and work of the National Committee on Mathematical Requirements. Each of these bodies has a distinct and important service to perform. The National Council is the youngest of the three and as yet the least closely organized, though its present membership of forty-three hundred is more than the combined membership of the other two bodies. Definite steps have just been taken at this Boston meeting which will doubtless tend to develop in the National Council a stronger spirit of group consciousness, of group enthusiasm and of group pride, all of which are most essential elements in any such organization. Some of these steps are: (1) the publication early next autumn of a Register of Members, (2) the revision and great enrichment of the Constitution, (3) the incorporation of the Council which will give it dignity and authority to handle funds in the form of gifts or bequests, (4) a definite program looking to the organization of new Branches of the Council and the federation with it of all existing clubs and associations of elementary and secondary teachers of mathematics, so far as this may be possible, and finally (5) the increase in the membership to a minimum of ten thousand within the next two years.

To accomplish all these objectives, and especially the last one, there will be needed the individual cooperation of all present members. The dues remain at two dollars, which include a subscription to the official journal, the MATHEMATICS TEACHER. No other similar organization has such small dues and this is possible with the Council only on the faith and irrepressible optimism of the officers and editors that the membership will grow very rapidly and immediately. *Every teacher of secondary mathematics in America should be a member of the Council and a reader of the* MATHEMATICS TEACHER.

Two important things stand out in the work of the Council: (1) the Year Books which have already been published and (2) the organization a year ago of a Branch of the Council in Louisiana-Mississippi in cooperation with the Section of the Mathematical Association in those states. The Year Books represent a definite, concrete service to the secondary field and the cooperation between college and secondary teachers as exemplified in the Louisiana-Mississippi program is another real service to the cause. Other sections of the Mathematical Association are

considering the desirability of such cooperation and it is to be hoped that this spirit will continue to grow. The Mathematical Association maintains a similar relation on the one side to the American Mathematical Society and I am sure it will be pleased to foster such cooperation on the other side with the National Council, as indeed it has already done when it sponsored the National Committee whose membership was more than half from the secondary field.

If every representative here this evening will carry back to his home club or association this spirit of enthusiasm, cooperation, and determination to make the National Council the greatest federated body of teachers in this country, I am sure that most important results will follow and that mathematics will continue to be recognized as all-pervasive in its influence in present-day civilization as the sunshine is in the physical world.

The Association of Teachers of Mathematics in the Middle States and Maryland will hold its annual Spring meeting on Saturday, May 26th, in the Auditorium at the Lincoln School of Teachers College, Columbia University, 425 West 123d Street, New York City. There will be a morning, an afternoon, and an evening program with a luncheon at 12:30 and a dinner at 6:30 at The Men's Faculty Club of Columbia University, corner of 117th Street and Morningside Drive. An excellent program is being arranged and notices will be sent out to members throughout the East. The price of tickets for the luncheon is $0.75; for the dinner $1.35. Reservations for each or both should be sent to the Secretary, W. D. Reeve, Teachers College, 525 West 120th Street, New York City.

TIME IN RELATION TO MATHEMATICS

By DAVID EUGENE SMITH, LL.D.

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Reason for Considering Time.—Because time is constantly referred to by the world's great poets and because it has for thousands of years been the subject of speculation by the world's great philosophers is no sufficient reason for considering it with relation to mathematics. Since, however, time is infinite, and infinity plays a major rôle in all branches of our science, it is not without value to the teacher to spend a little of time itself in considering its relation to mathematics, for it is a mathematical certainty that the time we spend will not lessen in the least that which is to follow through the ages, and beyond.

The Poet's Concept.—The infinity of time has always been a theme for the poets, both those who write in rhyme and those who put their poetry in prosaic forms. Sir Thomas Browne, the author of the *Religio Medici,* referred to it when he said that "the created world is but a small parenthesis in eternity"—he might have said a monomial in an infinite series. Tennyson called up the same idea in his not uncommon haze of mysticism, when he wrote,

> In time there is no present,
> In eternity no future,
> In eternity no past;

and Petrarch put it into these striking lines:

> And Time's revolving wheels shall lose at last
> The speed that spins the future and the past;
> And, sovereign of an undisputed throne,
> Awful Eternity shall reign alone.

In a similar vein Carlyle, whom the poets would wish to rank as a philosopher, and the philosophers as a poet, but whom both would agree to rank as a striking word painter, says in his *Heroes and Hero Worship:*

"That great mystery of Time, were there no other; the illimitable, silent, never-resting thing called Time, rolling, rushing on,

swift, silent, like an all-embracing ocean-tide on which we and all the Universe swim like exhalations, like apparitions which *are*, and then *are not:* this is forever and literally a miracle; a thing to strike us dumb—for we have no word to speak about it.''

All this seems interesting but intangible, unrelated to as exact a science as mathematics.   This which Milton calls

> The never ending flight
> Of future days

seems a subject for the poet rather than the mathematician.   In this view, however, we are wrong, and the philosopher steps in to show us the way.

The Philosopher on Time and Mathematics.—It is in the works of the philosophers that we find precision of statement as to the relation of time to mathematics, and this relation is not without interest in the school—notably so as set forth in the works of Schopenhauer.   This great thinker has given us perhaps the most striking description of the parallelism between time and mathematical concepts to be found in all literature.   With respect to arithmetic he remarks, in substance:

''That arithmetic rests upon pure intuition of time is not so obvious as that geometry is based upon pure intuition of space, but it may be readily proved as follows: All counting consists in the repeated positing of unity, only in order to know how often it has been posited we mark it each time with a different word, these words being numerals.   But repetition is possible only through succession, and succession rests upon the immediate intuition of *time*, being intelligible only by means of this latter concept.   It follows that counting is possible only by means of time.   This dependence of counting upon time is evidenced by the fact that in all languages multiplication is expressed by 'times,' that is, by a concept of time, as when we say 'six times seven.' ''

It was this idea, no doubt, that led Sir William Rowan Hamilton to speak of algebra as ''the science of pure time.''   Why, then, should we try to secure in the mind of youth some kind of meaning of space, and yet neglect wholly the meaning of time?

Time a Dual of Space.—Schopenhauer's greatest contribution in this field was his expression of the duality of time and space,

and it is this expression that the expositor of things mathemati-
cal will find the most helpful of all the similar cases of parallel-
ism in the works of the philosophers. Substantially, the most
thought-provoking features of his dualism are as follows:

TIME

1. There is only one *time*.
All different times are parts
of it.

2. Different *times* are not
simultaneous but successive.

3. Everything in *time* may
be thought of as non-existent,
but not *time* itself.

4. *Time* has three divisions:
past, present, and future.

5. *Time* is infinitely divisi-
ble.

6. *Time* is homogeneous and
a continuum; that is, no part
is different from any other
part, nor is any part separated
from another part by some-
thing which is not *time*.

7. *Time* has no beginning
and no ending, but all begin-
ning and ending is in *time*.

8. *Time* makes counting pos-
sible.

9. *Rhythm* exists only in
time.

10. *Time* has no p e r m a-
nence, but passes the moment
it is present.

11. *Time* never rests.

12. Everything in *time* has
duration.

13. *Time* has no duration,
but all duration is in *time*.

SPACE

1. There is only one *space*.
All different spaces are parts
of it.

2. Different *spaces* are not
successive but simultaneous.

3. Everything in *space* may
be thought of as non-existent,
but not *space* itself.

4. *Space* has three dimen-   ✳
sions: l e n g t h, breadth, and
thickness.

5. *Space* is infinitely divisi-
ble.

6. *Space* is homogeneous and
a continuum; that is, no part
is different from any other
part, nor is any part separated
from another part by some-
thing which is not *space*.

7. *Space* has no beginning
and no ending, but all begin-
ning and ending is in *space*.

8. *Space* makes measuring
possible.

9. *Symmetry* exists only in
space.

10. *S p a c e* is permanent
through all time; it n e v e r
passes.

11. *Space* never moves.

12. Everything in *space* has
position.

13. *Space* has no motion,
but all motion is in *space*.

✳Space has 2 divisions { Here,
                        { Not here.

TIME	SPACE
14. Motion is possible only in *time*.	14. Motion is possible only in *space*.
15. *T i m e* is omnipresent. Each part of it is everywhere.	15. *Space* is eternal. Each part of it exists always.
16. In *time*, all things are successive.	16. In *space*, all things are simultaneous.
17. Each part of *time* contains all substance.	17. No part of *space* contains the same substance as another part.
18. The '*now*' is without duration.	18. The *point* is without extent.
19. *Time* is of itself empty and indeterminate.	19. *Space* is of itself empty and indeterminate.
20. *Time* makes arithmetic possible.	20. *Space* makes geometry possible.*

Pointing to Time.—We speak of tomorrow and of tomorrow's tomorrow, and of the tomorrow of that day, and so on through infinite time to come. We also speak of yesterday and of yesterday's yesterday, and so on through infinite time that is past. We think of a thousand miles towards some fixed star, and of the thousand beyond, and of other thousands, and so on in what we—hurled about as we are by divers motions at speeds barely conceivable, through what we call space—assert is in a fixed direction. There are elements that are common between infinite time to come, infinite distance, and infinite time that is past. There is one feature, however, that is not common—we can point towards the fixed star, or at least we can pretend to do so; but we cannot point to tomorrow, nor can we point to yesterday— to time to come and to time that has past. We measure time and we measure distance, but we are helpless when we attempt to find a direction by which to denote the whereabouts of the twenty-first century, or the nineteenth, or to say where is the age of Pericles or the imagined age of Utopia. In this, then, we are as helpless as when we attempt to point towards a fourth dimension, or to any of the infinite dimensions that algebra uses with such freedom and success.

* For a more complete list, see R. E. Moritz, *Memorabilia Mathematica*, New York, 1914, p. 347.

Time a Fourth Dimension.—Sir William Rowan Hamilton, in speaking of his contribution of quaternions, remarked:

"Time is said to have only one dimension, and space to have three dimensions. . . . The mathematical quaternion partakes of both these elements; in technical language it may be said to be 'time plus space,' or 'space plus time': and in this sense it has, or at least involves a reference to, *four dimensions.*"

The passage is suggestive, and various writers have expressed similar views. A line to the north has extent, and we can point to the north; a line to the east has extent, and we can point to the east; a line to the zenith has extent, and we can point to the zenith; and by means of these three directions, taken negatively as well as positively, we can determine any point in our space. But when we consider Time we have seen that, although it also has extent, we cannot tell where lies the future or where lies its negative, the past. We are like a being in Flatland who can point lengthwise and breadthwise but not thickwise; while we can point thickwise we are utterly at a loss when we come to point timewise. So Time, whether or not it be a fourth dimension of which we have only a slight conception, at any rate suggests some of the features of such a dimension, besides suggesting, as Schopenhauer has shown us, infinite space.

If so, does it curve through the fifth dimension? Pictured on a plane, is its outline simply a vast circle that returns into itself? If we continue to the positive infinity of Time, do we reach the negative infinity; and if we continue on, do we return to the present once more? Was Lucretius right when he said "volat hora per orbem" (Time flies in a circle)? Does time simply run an infinite cycle, ever repeating itself throughout some supertime?—and then what?

These questions imply that time is comparable to a line instead of being comparable to three-dimensional space, as Schopenhauer suggested; but we may have two straight lines which intersect, while maybe it is meaningless to us to think of two times intersecting. Yet time has so many attributes of a straight line as to cause us to hesitate and to ask ourselves if, after all, there is only one time. Why need there be any such limitation? Are we, with respect to time, like a Linelander with respect to his narrow universe?

What does all this mean in teaching? Nothing as a class ex-

ercise; nothing as a formal talk to students; nothing as a precise doctrine to be studied as we study quadratics or graphs; but it means much if, from thinking of the mystery of Time, even a little of the mystery of Space is taught with a new vision, and if something is seen in geometry other than so many exercises for a lesson. It is the attitude of mind of the teacher, the imagination, the soul, the knowledge that count. In comparison with these, all the science of education that can be imparted in the schools is a feather in the balance.

IMPORTANT NOTICE!

At the Boston meeting of the National Council of Teachers of Mathematics, Professor Slaught pointed out three things which he thought the Council lacked. He referred to these as "Group Consciousness," "Group Pride," and "Group Enthusiasm." If we can secure the first, the other two will probably follow. This is why we need to enlist the support of every teacher of mathematics in this country.

We now have practically 5,000 members and we have set ourselves the task of securing 10,000 members by 1930. This means that we must double our present membership. However, this can easily be done if each teacher who is now a member will secure one other. From letters which reach us daily we know that there are still large numbers of teachers who know little or nothing of the work of the Council. We have accordingly published 20,000 circulars advertising the MATHEMATICS TEACHER and the yearbooks, copies of which we shall be glad to send to teachers who will agree to use them in situations where they may do good.

Finally, the Council will shortly issue as a supplement to one of the regular numbers of the MATHEMATICS TEACHER a register of members. It is important that each member of the Council send in at once his name, address, and teaching position as he wishes it to appear in the register. Return post-cards for this purpose will be found in this number. Otherwise the name and address of each will appear as it occurs on the mailing list.

THE EDITORS

MATHEMATICS IN MODERN BUSINESS [1]

By EDITH CLARKE

*Central Station Engineering Department, General Electric Company,
Schenectady, N. Y.*

I consider it a great opportunity to be allowed to speak to you
to-day. The subject assigned to me is Mathematics in Modern
Business, as exemplified in a large corporation such as the Gen-
eral Electric Company. So, first, I shall try to tell you some of
the ways in which mathematics is used to solve our technical
problems. After that, if you will allow me, I would like to tell
you how you can increase this use.

The General Electric Company is a manufacturing company,
making all kinds of electrical equipment. There are two general
classes of articles, first, those which are produced in quantity,
such as lamps, toasters, ranges, etc. These products are con-
tinually being improved and new standards developed, yet at any
one time the problem is one of production. Then there are the
large machines, such as steam turbines or generators which are
rarely made two alike. Of course, several machines which are
exact duplicates may be ordered at the same time for the same
power station, but seldom is a large machine duplicated at a
later date. The requirements of the next customer probably will
be quite different; and since improvements are constantly being
made he will select the machine which is the most economical.
This means that practically every large machine requires a spe-
cial design.

In addition to designing and manufacturing electrical appa-
ratus, we have consulting engineers who confer with the customer
and advise him in regard to his special requirements; also, we
have a large research laboratory where many scientists are at
work. The results of their work may, or may not, have a direct
practical application.

Since my experience has been largely in problems dealing with
the generation and transmission of electrical power, I shall give
you some examples of how mathematics is used in these problems.

A magnet will affect a compass needle, causing it to set itself

[1] Read at the Boston Meeting of the National Council of Teachers of
Mathematics.

along the flux line which passes through the point where it is (a short needle being assumed). Flux lines are closed loops which run from the north pole to the south pole of the magnet in air and from the south pole to the north pole within the magnet. They are imaginary lines invented to explain observed phenomena. If a conductor, such as a piece of copper wire, is moved so as to cut the flux lines of the magnet, a voltage will be generated in the wire, and if the wire is made into a closed circuit a current will flow in the wire. The voltage generated in the wire depends upon the strength of the magnetic field through which the conductor is moved, the length of the conductor and the rate at which it is moved. The current flowing in the wire depends upon the voltage which is generated in the wire and upon the impedance offered by the wire to the flow of current. To generate a high voltage we need a powerful magnet, a long conductor and rapid motion. For the powerful magnet, electro-magnets are used; that is, coils of wire carrying current. By placing an iron core in the coil, the magnetic field is strengthened. To make a long conductor, many coils of wire are used, sometimes with many turns per coil. To produce the rapid motion, a prime mover is required. This may be a steam turbine, a water turbine, a mercury turbine, a steam engine, gas engine, or oil engine. In a steam turbine, the steam is made to strike against the buckets which are on the turbine wheels. The buckets move, causing the wheels to revolve. As the wheels go around, the shaft turns with them. On the same shaft is placed the rotor, the moving part of the generator, and on this rotor the powerful electro-magnets. In the stator, the stationary part of the generator, are placed the coils which make up the conductors. These conductors may be connected through a transformer to a transmission line which runs for miles through the country, then through another transformer to a substation where power is supplied to a city such as this.

I have some slides which show steam turbines with their wheels and buckets; generators with their magnets and conductors; transformers, transmission lines and examples of some of the mathematics used in solving problems dealing with generation and transmission of electrical power.

1. Drawing of 165,000-kw. Steam Turbine in 3 units, for Philo Plant A. G. & E.—largest ever built. Longest bucket 35 in. Tip velocity 1100 ft./sec. 600 lb./sq. in.

2. Steam Plant, Buffalo General Electric.
3. 40,000-kw. Steam Turbine for Philo—A. G. & E.   600 lb./sq.
   in.   19 stages.
4. 10,000-kw. steam turbine for Edgar Station, Boston.   16
   stages.   18 in. diam.   3 feet long.   1200 lb./sq. in.
   Buckets 1.2 in. to 2.8 in.
5. Stator of small generator.
6. Rotor of small generator.
7. ½ of stator of large generator.
8. Stator frame 37 feet in diameter.   Conowingo gen. 40,000-kv.
   88 poles, 81.8 r.p.m., 13,800 volts.
9. Rotor, Conowingo.
10. Rotor.
11. Generators at Muscle Shoals.
12. Shoshone Falls—Idaho Power Co.
13. Lock 12—Alabama Power Co.
14. Keokuk—Mississippi Power Co.
15. Transmission Line—Southern California.
16. N.E. Power Co. Station, Outdoor Sub.
17. Transformer.
18. Wheel Section:

The last is the cross-section of a disc wheel of a steam turbine
with one bucket.   The bucket is so placed that when the steam
strikes against it the wheel will revolve.   There will be stresses in
the wheel due to centrifugal force.   Stodola, in his book *Steam
and Gas Turbines*, has worked up the case for a disc wheel where
the outline can be represented by one smooth curve.   In this
work, Stodola has solved the differential equations involving the
stresses, the elasticity of the material, the elongation and contrac-
tion due to stress.   The mathematics and physics are so related
that to understand one it is necessary to know both.   Mr. Weaver,
of our company, has extended Stodola's work to the case where
the outline of the disc wheel is not a smooth curve, but can be
represented by several curves, as shown on the slide.   At the
present, although the work is long and involved, the various proc-
esses have been so systematized that they can be handled very
efficiently by high-school girls.

About eight years ago, it was decided to calculate and record
the stresses in every wheel of every turbine that had ever been

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DISC WHEEL STRESS
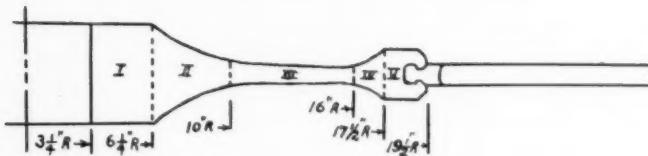
The Wheel Section is divided into a number of Rings
So that the outline of each Ring is a smooth Curve

Equation of outline:

$$\frac{t_2}{t_1} = \left(\frac{r_2}{r_1}\right)^a \qquad \text{Where } t = \text{thickness}$$
$$r = \text{radius}$$

Equations for Stresses:

$$T_1 = A r_2^2 - B R + C R_2$$
$$T_2 = D r_2^2 - E R_1 + F R_2$$

Where T = Tangential Stress
R = Radial Stress
A, B, C, D, E, F — Constants from Charts

built. I was given the job of selecting the girls to do this work. It may interest you to see the following test which I gave them. Probably it looks ridiculously easy to you. I felt that if

TEST SLIDE No. 19

1. $2.5 \times 0.020 =$
2. $0.03 \div 0.00015 =$

3. $B = x - (a/2)$
   Find B when $x = 4.2$
   $a = -2.6$

they understood decimals and the use of the negative sign the rest could be learned. Practically every one had the correct answer to the first problem, but almost no one gave the second answer correctly immediately. The girls I chose were the ones who proved their answer. This answer was correct and they were *sure* of it. Anyone who can multiply should be able to give the correct answer to a problem in division, even if she does not know how to divide. The most used implement of the engineer is the "cut-and-try method." A value is assumed. Under the given conditions, does this value give the observed results? The girl who guessed her answer and then proved it was using this method.

20. Steam Chart:

This chart is plotted from tests, extended somewhat beyond the range covered by the tests. This work on steam properly belongs to the scientists and is being done by them. The steam tables of Marks & Davis are used for low pressures and temperatures up to 300° F. The work of Davis & Kleinschmidt of Harvard going to 565 lb. abs. and 650° F., completed about four years ago, is included in this chart, which will be extended to include the work on specific volume recently done at M. I. T. The research work in the colleges is being done under the Steam Research Program of the A. S. M. E., otherwise it would have to be done in the General Electric Company.

21. Testing Wheel:

This wheel is being tested for vibrations. About eight years ago we had trouble with turbine wheels. By testing many wheels it was found that the trouble was due to vibration and could be remedied by changing the distribution of the material in the wheel. We know now how to build wheels to avoid this trouble. This is an example of a problem which has been solved by tests but not yet by mathematics.

22. Transmission Line Calculations:

The only kinds of engineering problems which are purely mathematical are calculations. Sometimes a great deal of time

Transmission Line Calculations

$$e_g = e_r \cosh \sqrt{ZY} + i_r \sqrt{\frac{Z}{Y}} \sinh \sqrt{ZY}$$

where

$$Z = R + j\, 2\pi f L = z\ell\, \underline{|\theta_j}$$
$$Y = G + j\, 2\pi f C = y\ell\, \underline{|90°} \quad (G = 0)$$

Since

$$\cosh \sqrt{ZY} = 1 + \frac{ZY}{\underline{|2}} + \frac{Z^2Y^2}{\underline{|4}} + \frac{Z^3Y^3}{\underline{|6}} + \cdots$$

$$\sqrt{\frac{Z}{Y}} \sinh \sqrt{ZY} = Z\left(1 + \frac{ZY}{\underline{|3}} + \frac{Z^2Y^2}{\underline{|5}} + \cdots\right) = z\ell\beta\, \underline{|\theta_j + \theta_0}$$

$$\frac{i_r}{e_r} = \frac{KW.}{\sqrt{3}\,KV.(p.f)\,e_r} = \frac{KW\, 10^{-3}}{KV^2(p.f.)} = \frac{K}{p.f.}\,10^{-3}$$

$$\therefore \frac{e_g}{e_r} = 1 + \frac{ZY}{2} + \frac{Z^2Y^2}{24} + \cdots + \frac{K}{p.f.}\,z\ell\beta\, 10^{-3}\, \underline{|\theta + \theta_0 + \theta}$$
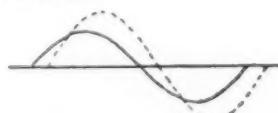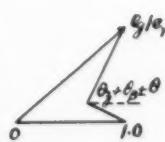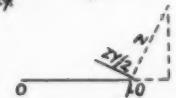
may be saved by putting the given equations into a different form. This slide shows how the voltage equation for a transmission line was put into such form that the solution could be obtained graphically by means of a calculating device.

23. Calculating device—The Transmission Line Calculator.

24. Sparking at Contacts of Voltage Regulator:

By the use of mathematics, as indicated on this slide, it was shown that to avoid sparking the contacts must spring apart at the first instant. After a spring was put in, the current which

Sparking at Contacts of Voltage Regulator
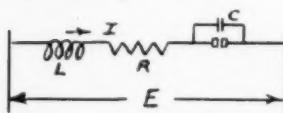
$$L \frac{di}{dt} + Ri + \frac{q}{C} = E$$

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E$$

$$q = A + A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}, \quad \text{where} \quad \alpha = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$e = \frac{q}{C} = E + K_1 e^{-\alpha_1 t} + K_2 e^{-\alpha_2 t}$$

$x = $ distance contacts open $=$ velocity $\times$ time $= vt$

$\frac{e}{x} = \frac{e}{vt} = $ volts per cm. $=$ Potential Gradient

$\frac{e}{vt}$ is maximum at first instant, since $t = 0$ when $\frac{d(e/vt)}{dt} = 0$

To avoid sparking contacts must open with velocity, v where v is obtained from the equation

$$\frac{e}{vt} = 10,000 \quad , \quad \text{at time } t = 0$$

could be interrupted without sparking was increased from 1 ampere to 20 amperes.

26. Geometry:

Here are some problems in geometry which have come up in connection with engineering problems.

27. Field Current on Short-Circuited Generator:

This slide was taken from a paper by Mr. R. E. Doherty and Mr. C. A. Nickle on Synchronous Machines presented to the American Institute of Electrical Engineers. It shows the actual current as obtained by an oscillogram and the calculated current.

Geometry
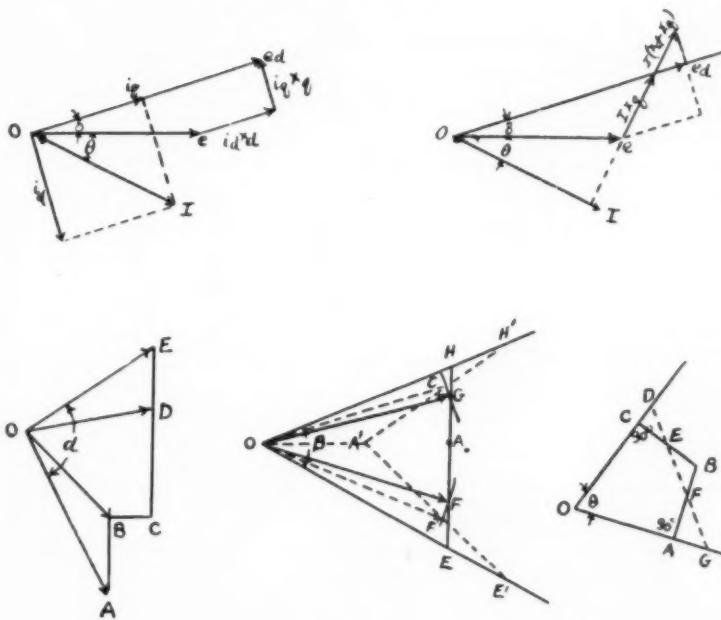
You see how perfectly the calculated values check the test values. I give it as an example of a correct solution to a most difficult problem made possible by mathematics combined with practical knowledge.

Very rarely do we have an engineering problem which is a mathematical problem—pure and simple. If this were so, we could hand our difficult problems over to the mathematician and he could give us the solution. The thinking has to be done in the physics of the problem and if the engineer can put this into mathematical terms he can usually solve it himself. The hard part is making the step from physical relations to mathematical relations. To build up the theory from the beginning in any particular line requires mathematical ability and training of a high order, as well as a thorough understanding of the physical relations involved.

Why is it that so few of us can do this? I think it is due to the way our mathematics was given to us—always as an end in itself, as a thing apart. You note I use the past tense. From

the exhibits around this room and the work which is being done in some of the schools I am led to hope that conditions now are changing.   If we are to be able to use our mathematics afterward, in solving physical problems, we should have practice at an early date in expressing physical relations in simple mathematical terms, these terms becoming more complicated as we progress. Studying mathematics without practical application is like study-ing the piano with no piano in sight, but an occasional inspection trip to one and possibly to touch it once or twice.   It is the ex-ceptional person who could play the piano after such a course, but I think there might possibly be a few who could, even so handicapped.   The engineer who can go with ease from the physical to the mathematical and back to the physical is just as exceptional.

It is necessary to be expert in the mechanical operations, such as solving equations, passing from one trigonometric identity to another, in seeing and proving geometric relations, in finding derivatives and performing integrations, but all these operations could be made so much more vital to us if they had a direct practical application, if the problems were given first and the solutions as needed.

Would it not be possible to combine geometry with surveying or carpentry, or dress-making? I know a boy who graduated from high school well up in his class, but when his brother, who was doing some carpentry work, asked him to lay out a square corner, he could not do it.   If he had been asked to prove that the square inscribed on the hypotenuse of a right angle triangle is equal to the sum of the squares on the other two sides, he could have given a satisfactory demonstration.

Four years ago last fall Mr. R. E. Doherty, one of our consult-ing engineers, started a course of study in the General Electric Company, the object of this course being to train engineers to build up theory, starting with fundamentals.   It is a three years' course open to young engineers in the company.   Entrance is by competitive examination.   Two classes have completed the three years and men from these classes are among the outstanding young men of the company.

I talked to the foremost engineers and scientists in our com-pany in regard to the contribution of the mathematician to the advancement of engineering and science.   Their views are so

different that I shall try to give them both to you.   The engineer says: "There are many technical problems waiting to be solved which require the ability and skill of the mathematician.   If he would cooperate he could help increase our technical efficiency, thereby adding to the prosperity of the country.   Isn't it selfish of him to be spending his time on some investigation which may or may not be of any use?   It is like working on a puzzle or a game of chess which tests his ingenuity to the utmost but which gives no practical results."

The scientist says: "The pure mathematician has done wonders for the advancement of science.   He works on his problem without regard to the use which is going to be made of his results but in many cases he anticipates the needs of the physicists.   Take for example integral equations or matrix algebra given by mathematicians without thought of their use.   They are essential for the Quantum Theorem.   Born has developed matrix calculus, differential and integral equations, but the framework was there for his use.   Of course a man should not spend all of his life working on a puzzle or playing chess but if he could discover a general principle underlying all puzzles or work out a scheme whereby he could always win a game of chess, he would be making a tremendous contribution.   It is opening up new lines of thought that we need and the pure mathematician is doing that." So the choice is with you to cooperate with the engineer or to point out the way to the scientist.

I should like to leave this thought with you: If you teach mathematics, let your students see it as something essential to the solution of problems which they really want to solve.   Leave the choice to them as to whether they keep it pure or make it efficient, but make it possible for them to apply it if they wish to do so.

APPLICATIONS OF INDETERMINATE EQUATIONS TO GEOMETRY

By M. O. TRIPP

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One of the important problems in secondary instruction in mathematics is to find concrete material for vitalizing the work in elementary algebra and geometry. Frequently the work as given is too abstract for the student, and does not connect closely with his experience. Since the road to the abstract is over the concrete, it is hoped that the illustrative material here offered will make the student's approach to generalized mathematics easier by using much arithmetical work.

In studying the rule for the square of the sum of two numbers and the square of the difference of two numbers we may construct a right triangle with sides as shown in the figure. The student
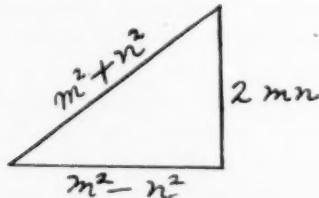
can readily verify, in this representation, that the square on the hypotenuse is equal to the sum of the squares of the two legs, that is,

$$(m^2 + n^2)^2 = (2mn)^2 + (m^2 - n^2)^2.$$

The connection between algebra and arithmetic may be brought out by having the student substitute different numbers for the letters, thus getting various triples of rational numbers such that the sum of the squares of the two smaller numbers equals the square of the largest of the triple. In doing this it is evident that m must not be taken equal to n, for then the triangle ceases to exist. Moreover, to avoid negative numbers, m must be taken greater than n. Thus, if $m = 2$ and $n = 1$, we have a right triangle with the sides 3, 4, 5.

We may now define a set of Pythagorean numbers as any triple of rational numbers of such a nature that they may be taken to represent the three sides of a right triangle. In general, it is best to limit ourselves, in this case, to rational positive integers, such as the triple 3, 4, 5, above.

Equimultiples of 3, 4, 5, such as 12, 16, 20, furnish triangles similar to the first right triangle. In order to obtain triples of numbers such that none of the triangles shall be similar we must take m and n relatively prime. *& not both odd.*

The above relation between m and n will be found very helpful in preparing problems on the right triangle, such as the ladder exercises. By testing numbers with the Pythagorean equation

$$(m^2 + n^2)^2 = (2mn)^2 + (m^2 - n^2)^2,$$

one can make sure that when two sides of the right triangle are given the third will come out rational, which is a decided advantage in elementary work.

The determination of a rational triple of numbers a, b, c, which will render A rational in the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

may be carried out by the use of the relation

$$(m^2 + n^2)^2 = (2mn)^2 + (m^2 - n^2)^2.$$

We now define a triple of Heronian numbers as any set of three rational numbers, representing the sides of a triangle, such that the area is a rational number. Thus, if $a = 13$, $b = 14$, $c = 15$ be the sides of a triangle, we get, by the use of the formula for A above, the rational number 84 representing the area of the triangle.

In order to determine further Heronian triples we consider an oblique triangle broken up into two right triangles as in the figure.
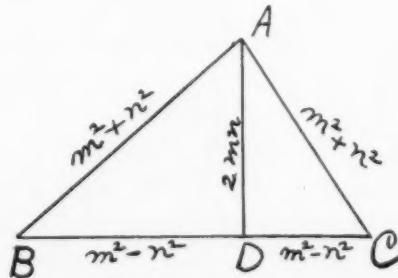
We now choose two pairs of values for m and n so that the product $2mn$ shall be the same for each pair; and then we use one pair of values for the triangle ABD and the other for the triangle ACD.

As a concrete illustration let us take $2mn = 12$, leading to the two pairs of values $m = 3$, $n = 2$; $m = 6$, $n = 1$. By using the first pair for the triangle ABD we have $AB = 13$, $BD = 5$. Using the second pair for the triangle ADC we have $AC = 37$, $DC = 35$. Hence we have the triple of Heronian numbers 40, 13, 37.

The teacher who has at hand triples of Heronian numbers can make up problems in simple equations on the areas of triangles. Thus we may find the area of the triangle whose sides are 13, 14, 15 as follows: In the above triangle let $BC = 14$, $CA = 13$, $BA = 15$.

Let $x =$ the distance BD in the figure. Then $14 - x =$ the distance DC in the figure. By using the two right triangles BDA and DAC we have

$$15^2 - x^2 = \overline{DA}^2 = 13^2 - (14 - x)^2,$$

or

$$225 - x^2 = 169 - 196 + 28x - x^2.$$
$$\therefore \quad 252 = 28x,$$
$$9 = x.$$

Since $BD = 9$ we have $AD = 12$. Accordingly the area of the triangle is

$$\frac{14 \cdot 12}{2} = 84.$$

The problem of constructing regular polygons offers an interesting application of indeterminate equations. It is well known that certain series of regular polygons can be constructed with the Euclidean instruments, rules and compasses; while other series of regular polygons are non-constructible under these conditions. Thus we have as an example of constructible regular polygons

$$3 \cdot 2^n \quad (n = 0, 1, 2, 3, \ldots),$$

while an example of a non-constructible series is

$$7 \cdot 2^n \quad (n = 0, 1, 2, 3, \ldots).$$

From the fact that a regular triangle and regular pentagon can be constructed we may determine the method of inscribing in a circle a regular polygon of fifteen sides by means of the indeterminate equation

$$x/3 + y/5 = 1/15,$$

where x and y are positive or negative integers. The unit involved in this equation is the perigon, or four right angles. The term $x/3$ means that one third of a perigon may be taken additively or subtractively x times, an operation that can be carried out by the methods of elementary geometry. Likewise $y/5$ means that one fifth of a perigon may be taken in the same manner.

If integral solutions for x and y can be found, then one fifteenth of a perigon can be constructed, or a regular polygon of fifteen sides can be inscribed in a circle. Other cases similar to the above may be found in the Chapter on Regular Polygons in Schultze's *Teaching of Mathematics in Secondary Schools.*

In order to find the integral solutions of the above equations we first clear of fractions.

$$\therefore \quad 5x + 3y = 1.$$

In order to solve this equation the method given by Fine, *College Algebra*, pp. 343–4, will be followed.

Solving for the variable with the smaller coefficient we have

$$y = (1 - 5x)/3 = -x + (1 - 2x)/3.$$

Hence for integral values of x and y the fraction $(1 - 2x)/3$ must be an integer.

Let

$$u = (1 - 2x)/3.$$

Then

$$3u + 2x = 1.$$

Solving this equation for the variable with the smaller coefficient,

$$x = (1 - 3u)/2 = -u + (1 - u)/2.$$

Since x and u are to be integral,

$$(1 - u)/2$$

must be an integer. Clearly this will be true if $u = 1$.

But if $u = 1$, then $x = -1$; and if $x = -1$ in the original equation, then $y = 2$.

The interpretation of this solution is that we construct two times one fifth of a perigon, viz., 144°; and then subtract from this one third of a perigon, or 120°. We thus get an angle

$$144° - 120° = 24°,$$

which is one fifteenth of a perigon.

By considering regular polygons of five sides and seventeen sides we may construct a regular polygon of eighty-five sides.

The equations of the type we have been considering always have integral solutions, and hence the regular polygons of the type considered are always constructible.

MATHEMATICS IN SCIENCE [1]

By H. W. TYLER

Massachusetts Institute of Technology, Boston, Mass.

Introduction

I count it an honor, however little deserved, to have a responsible share in your program. When I received the invitation from the President *in esse* I had not quite the strength of mind either to decline or to admit that the subject was too big for me. The President *in posse* in a flank attack made a specious plea about the duty of the head of a department in an institution reputed to be a hotbed of applied mathematics to justify that reputation —a plea which it would have been ignominious to disregard. Aware, nevertheless, of my real incompetence to tell you just what part our ancient science really does play in these other more complicated but less interesting modern ones, I have, after trying to show some of you, yesterday, a few exhibits of transmigrated mathematics, surreptitiously changed my subject without the permission of either of these presidential potentates to *Mathematics as a Modern Science*, naturally from the standpoint of a teacher in a school of applied science.

In regard to these other sciences and near-sciences, I will merely remark in passing—without troubling you with unnecessary proof—that their claim to be called science is approximately proportionate to the extent to which they have become mathematical. Some of the younger among them are still in the quite cartilaginous stages of flapperdom, with that limited respect for their elders which too often goes with such immaturity. After all, my title doesn't really matter so very much, because my treatment is likely to be too discursive and my selection of topics too heterogeneous to correspond with any one title however elastic and comprehensive. If I should be careless enough, or foolish enough, to stray inadvertently into the fields of your other speakers, I feel sure they won't mind the chance to expose me, refut-

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[1] An address at the Boston Meeting of the National Council of Teachers of Mathematics, Feb. 25, 1928.

ing or improving my presentation. I shall take particular care *not* to trench on the province of the accomplished speaker who is to follow me, and shall gladly leave her to tell you all about the mathematical side both of modern business and of modern invention.

My first sub-title shall be:

Science and Education

Quoting from a distinguished mathematical physicist: "We are apt to slip into the absurd error of supposing that there was practically no science before the nineteenth century. . . . It is in the popular appreciation of science rather than in science itself that the last century has proved absolutely revolutionary. . . . This revolution has been brought about, not so much by scientific discoveries themselves as by the application of these discoveries . . . inventions of all kinds that have completely changed the conditions of our daily life.'' To-day science claims the whole field for its parish—and seeks to make its influence predominant in the world of business and of government. This emphasis on the social aspect of science and scientific training holds up social effectiveness, power to serve the community, as the end of education. It asks not so much what you know as what you can do. We are still too much under the influence of the classical tradition. Science prizes facts so highly that its teachers are prone to be content when these facts are learned. It is just this that causes disappointment with science as a means of education. It is infinitely harder to instil a scientific spirit into a boy through the medium of chemistry and make him thereby a more useful citizen than to rub in a few facts as to the constitution of water or the preparation of chlorine.

Inherited Mathematics

One of the most obvious and significant facts about mathematics is its antiquity. This, combined with what I may perhaps call its highly consequential character, which makes most of its new developments so difficult of access, has given it in the popular mind a quite misleading appearance of rigid immobility, as of a fossil or mummy long since past the possibility of growth.

The subject-matter of school mathematics varies in age from a thousand years upward, many of its problems are antediluvian.

even though sometimes ineffectually veneered with up-to-date phraseology. Most college mathematics is at least as old as Euler. Largely in consequence of its age and the earlier development of its pedagogy and texts, mathematics has long been a highly respected and protected subject in the school curriculum, in the requirements for admission to college and in the work after admission. It has shared with the classics the reputation of being not a mere means to practical ends but an exceptionally nutritious brain-food.

The reaction of this artificial protection—as of similar conditions in other fields—has inevitably been harmful if not indeed disastrous. The new, unprivileged subjects, like the proverbial poor boy, have had to fight their way into the curriculum on merit—real or alleged; the vested interests of classics and mathematics have had to choose between the alternatives of bending and breaking, of reconstruction and elimination.

Progress in Texts

The teachers and text-book writers have had to discontinue the habit of repeating each others' mistakes, in addition to their own, of perpetuating relatively useless material and antiquated pedantic methods of presentation. How much they have accomplished in this progress many of you know far better than I. I have lately learned it in part by perusal of the admirable booklets on progress in arithmetic and algebra by our learned friend, Professor D. E. Smith.

Report of the National Committee

Not having been called to order yet for my digressions, I yield to the temptation to interpolate a word about this. As a sort of supernumerary member of the committee I must not speak too kindly of it and of course I am not inclined to extreme severity. Some critics, I believe, have chided the committee and depreciated its report for dealing too much with opinions rather than with facts established by analytical research. At the risk of exposing one of the numerous infirmities of age—more colloquially of admitting myself an unreformed "old fogy"—I venture to inquire how completely it is possible to dissociate facts and opinions, or judgments, and with what certainty of advantage. If one has to choose between judgment without statistics and statistics without

judgment, which is safer? Isn't judgment based on prolonged experience itself a fact of high order? Is a fact only then a fact when it is measurable, and are we quite sure that the highest values don't transcend our still immature metrical devices? Do we estimate the value of a sonnet by counting the letters in it? Please do not infer that I am opposed to modern experimentation, new type examinations, psychological tests and the like. I would only, with all deference, suggest that a certain reticent humility would sometimes be in place.

MATHEMATICS TEACHERS

It must be admitted that just as mathematics has suffered by being so long a protected industry, so also have mathematicians sometimes failed by the narrow concentration of interest within their specialty—a concentration which within due limits is a necessary mathematical virtue. Fraternizing with each other, we have been too heedless of the enemy at the gates; too negligent of our duty to convince the plain citizen and the educational administrator of the impossibility of building an enduring educational structure without a due share of mathematics in its foundation; too reluctant to mend our mathematical ways rather than to see them ended.

Mathematics as a Tool

(This I think tends toward the subject assigned me.) It was the fashion with a certain critical group a few years ago to insist that mathematics and mathematicians must be put in their places of due subordination under the slogan, "Mathematics as a Tool." The question then arises: What is a tool? My nearest dictionaries say "an instrument of manual operation"—not mental you observe—but also figuratively a thing used in an occupation or pursuit, as literary tools, the tools of one's trade. In this broader sense it would seem that the English language is as truly a tool as mathematics. In the sense of the critics it was sometimes implied that one should use mathematics much like a table of logarithms or a computing machine, feeding in data at the hopper and grinding out results at the other end. The only justification for so superficial a view would be one based on rigorous time conditions. Does the thorough study of mathematical principles involve an excessive expenditure of time for

the students in question? Here, as so often, the sane teacher avoids both extremes. There must be no attempt to make professional mathematicians. Such attempts have of course been very rare. They are born, not made. The subject must be presented with strict economy of time, with resolute avoidance of unnecessary complications and non-essential material, still lagging superfluous on the text-book stage. This has been a hard lesson for the mathematician to learn, and too often the question has been begged by teaching nothing better than the use of a handbook. We stand in the Institute of Technology, so far as I can speak for it, for strong and continued emphasis on essential mathematical principles, for mathematics taught by mathematicians, with emphasis on proofs and theory, but not on those refinements of rigor which would have no clear meaning for freshmen and sophomores. Our mathematics is certainly a tool, but to make it a really efficient tool it must become an ingrained habit of analytical thinking and not a mere bag of tricks of computation—a tool of shoddy alloy.

Unified Mathematics

It is twenty years since we adopted the principle of merging calculus, analytic geometry and the elements of differential equations in a continuous program; ten years since we took the further step of starting our entering freshmen at once in elementary calculus. The relation of this to Mathematics in Science may need a word of explanation. Mathematics below calculus is mostly static; with just caution it ventures only on "average speed" with the unfortunate result that the weaker students cling ever after with desperate tenacity to the inadequate school formula $V = s/t$, stoutly resisting the real calculus fact $V = ds/dt$.

Is calculus too difficult for freshmen or even for superior high school seniors? Is it worth while in a world continually changing to stop short of the discussion of rate of change? Is the automobile trap unintelligible? I like to tell my students that the court will not accept $V = s/t$ as a defence in a case of overspeeding; that the automobile trap for obvious reasons uses $\Delta s/\Delta t$ and that our definition of derivative as speed merely completes the transition to the limit. If a student can explain instantaneous speed of a moving particle or steepness of a curve at

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a point, he has overcome the initial difficulty of calculus and probably improved his command of the English language. For purely mathematical purposes he might precede this accomplishment by an elaborate course in analytic geometry. For making his mathematics mean something in his physics, however, it is a great advantage to get this elementary calculus and more like it early in the freshman year. So Mathematics in Science means easy calculus early and a cumulative repeated emphasis on the calculus method of dealing with continuously changing quantities whenever and wherever they occur.

In my student time differential equations was a remote and little explored region—at any rate for students of chemistry— but of course that was a very long time ago. Now it is a very short step indeed from $V = ds/dt = c$, to $d^2s/dt^2 = c$, and even $d^2s/dt^2 = k^2s$. Of course there are many desert miles of technique to be traversed on the way but something has been done by way of irrigation and elimination of detours, and the early work has made objectives more or less real and attractive. In our present program a student of exceptional capacity may even gain so much mathematical perspective from his first semester's introductory course in derivatives and integration of polynomials that he can telescope the later program to an extent formerly unthinkable.

Economy of Time

I dimly remember studying college algebra and trigonometry in my first student year in the Institute, analytic geometry and differential calculus in the second, integral calculus in the first half of the third, and no more—being indeed a chemist. All, or nearly all, applications were geometrical, the subject was complete in itself, most of it quite unrelated to my chemistry and physics—a more or less fascinating recreation. Now the relatively few essentials of the algebra and trigonometry are required for admission or taught as needed; analytic geometry is merely a method continually used in the calculus, applications to mechanics are frequent. Every effort is made to facilitate the always difficult transition from the abstract to the concrete, to wean the student from his rooted antipathy for making analysis and geometry help each other. Every other scientific or professional department recognizes the value of the mathematical foundation and method—even to the chemists and biologists who

formerly fought shy but now find their sciences becoming quantitative, therefore dependent on mathematics. The electrical engineers and the physicists expect most from us but some of the others are not far behind. Of course they all expect us to do our work in the least time we can; of course we are occasionally judged, or misjudged, by the sometimes appalling errors of students who have "got by," much as we berate the schools for the sins of our freshmen.

Some Conclusions

The conclusion of the whole matter with particular reference to the school curriculum seems to me therefore something like this. Without sound and adequate training well started in youth, mathematics will be for most people practically unattainable. This training should include a first course in calculus, whenever early school work has indicated reasonable capacity. It is entirely feasible to reach elementary calculus with college freshmen provided the mathematical program is properly simplified without undue sacrifice of other subjects. Such a plan, aside from relatively infrequent cases of marked inaptitude, should be an important element in the education of a scientific worker in any field. Capacity for a scientific career is so important that no boy or girl should be prematurely barred by the exclusion or discouragement of what I believe are called exploratory courses in the elementary mathematics. Capacity for it rather than incapacity should be presupposed. On the other hand, we mathematicians should recognize and meet our obligation to justify our work and our plans, reforming our texts and our methods, discarding the obsolete and the non-essential, simplifying technique and vocabulary, eliminating pedantry. So we shall render our science secure in its eternal position in modern as in ancient science.

GEOMETRY NOTES [1]

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These notes offer clearer statements of some propositions of plane geometry and more consistent treatments of others. They would put some propositions from required lists to exercises. And they look at "rotation-translation."

For the data more than seventy American secondary school texts in geometry were examined, fifteen junior high school texts, some reports, and some foreign texts. The list is far from exhaustive but the sampling is representative.

The National Committee advises ". . . mathematics may be and should be a genuine help toward the acquisition of good habits in the speaking and writing of good English." Noting this advice, let us consider the wording of this proposition: "If two sects drawn from a point in a perpendicular to a line *cut off equal distances from the foot of the perpendicular*, they are equal." "Cut off from . . ." suggests that an operation is being performed on the foot of the perpendicular. At best the statement of the proposition is most awkward. Some authors aim to avoid ambiguity by writing ". . . cut off on the line equal distances from the foot of the perpendicular. . . ." This protects the foot of the perpendicular but is almost as awkward as its original.

Neither of the forms just quoted appears to have been used before Chauvenet in 1869. Before that time Loomis, Davies' Legendre, Robinson, Greenleaf, made the sects "meet the second line at equal distances from the foot of the perpendicular. . . ." This is entirely clear and is used in many texts. It is adopted in the Chauvenet revised by Byerly.

The following is a concise statement of the proposition:

"If two sects from a point to a line make equal projections on the line, the sects equal each other."

[1] Read at the twenty-fifth annual meeting of the Association of Teachers of Mathematics in New England in Boston, December 3, 1927.

In recent texts as well as in older we read: "If *three* sides of one triangle equal respectively *three* sides of another, the triangles equal each the other." Twelve books in the number of those consulted state the proposition in this way.

The definite article *the* has a certain function. The omission of this article before *three* in the proposition allows it to be understood that the triangle has more sides than three and makes the statement faulty. It is probable that the fifty or more authors who use the article recognize the need when they write: "If the three sides of one triangle equal respectively the three sides of another, the triangles equal each the other."

While the preceding is a correct statement, its enunciation is often faulty. Students frequently omit the article. This is due to the halting sequence of aspirates in "If the three" and the student follows a line of less resistance by omitting what he considers of less importance. The remedy is simple. "Three" in this place is quite superfluous. "Triangle" tells the number of sides. The article is needed absolutely. Let the proposition read then: "If *the* sides of one triangle equal respectively *the* sides of another, the one triangle equals the other." If "three" is not mentioned in the proposition, it will be out of sight and out of mind and will not prompt inaccurate expressions.

"The sum of the angles of a triangle equals two right angles" is the common reading of this proposition. The absence of "three" after the article saves some of the faulty language which follows its use. Perhaps twelve books write "the three."

One text has these four propositions:

"If *three* sides of one triangle equal respectively *three* sides of another, the one triangle equals the other;"

"The sum of *the* angles of a triangle equals a straight angle;"

"If *the three* angles of one triangle equal respectively *the three* angles of another, the triangles are similar;"

"If *the* sides of one triangle parallel respectively *the* sides of another, the triangles are similar."

The first statement is incorrect. The second and the fourth are perfect. The third is correct but it would give better results without "three." The diversity here illustrated shows that writers often are too intent on content and forget the needed form.

Let us recall Euclid's bisection of a sect.

Given sect AB to be bisected. On AB as base let a regular triangle ABC be constructed. Let angle ACB be bisected. Euclid bisects by constructing a regular triangle ABD. The line CD is bisector of angle ACB and sect AB. We note that the circles of radius AB and centers A and B contain the vertices of the triangles.

Heath in the notes to his Greek edition of Book I of Euclid gives in full the bisection by Apollonius of Perga: Given sect AB to be bisected. Let circles be drawn of radius AB and A and B centers. Their intersection at C and D is assumed. CD is the bisector of AB. The construction is shown to be valid by completing the triangles as in Euclid. The approach in Apollonius differs from that in Euclid but the figures are identical and the proofs are essentially so.

Of American texts Halsted's Plane has the Euclidean proof. The Apollonian proof is given in Halsted's Synthetic, Beman-Smith, Gillet, Wentworth-Smith, Wentworth-Smith-Brown, Auerbach-Walsh, R. R. Smith, E. R. Smith. Three of these give alternative proofs.

What is the favorite American bisection as it appears in more than forty texts? This:

Required to bisect AB. Let A and B be centers and radius be greater than half of AB and let circles be drawn. These are assumed to intersect in C and D and CD is the bisector sought.

Let us state this proposition in another way.

Required to find a *half of AB*. With radius *greater than half of AB* and A and B centers let circles be drawn. And so on.

In Dodgson's "Euclid and his Modern Rivals" Minos and Herr Niemand discuss Legendre as one of the modern rivals. Minos speaks: "I have observed only one instance of faulty logic. It occurs in Problem I, where in order to bisect a given line we are told to assume a length 'greater than half of it.' This would appear to require the previous solution of the problem, and, therefore, is, strictly speaking, a *petitio principii*."

So, in a construction for proof of this proposition a radius not less than the sect to be bisected is required. A fraction of AB is not admissible. But forty or more of the books mentioned make the inadmissible assumption and lead their trusting young readers in their first steps in the practice of assuming that which is to be proved.

In this construction Euclid, Apollonius, and their modern followers are consistent. In addition there are some writers who apparently are conscious of the inconsistency of the "greater than half" procedure and maybe think "radius AB" pedantic. They try a middle course if such there be. Some of these use a "convenient radius," others "a sufficient radius." Some take "equal radii," some "the same radius." One uses "arcs intersecting above and below AB." One has "intersecting circles," another "radius great enough to give two points of intersection." And so on with minor variations. Most interesting are the following: "Take radius manifestly greater than half of AB," and "Take radius greater than the apparent length of half of AB." Of these substitutes "sufficient radius" is to be preferred because it warns the student that he is changing his mode of attack.

It may be argued that the necessary radius has to be greater than half of AB in any case because two circles the sum of whose radii is less than their centerjoin can not intersect. One writer establishes the conditions of intersection as part of his proof, as should be done. Even so the half is sought and can not be used as a datum.

This construction has had a vagarious course in one set of books. The first text published had "radius AB," the second "equal radii," the third had "radius AB" in the formal construction and "convenient radius" in the informal, the fourth has "convenient radius" in the formal construction and "radius AB" in the informal. And the latest has "a convenient radius" for the only construction, and this note: "A convenient radius in many cases is AB itself."

Young and Schwartz has this proposition: "An angle inscribed in a circle equals one half of the center angle subtended by the same arc." Seeking information about this we consult Euclid and find in Book III Proposition 20 which reads: "An angle at the center of a circle is double of the angle at the circumference . . . on the same arc." Each of these forms of the proposition is a direct and simple statement of the fact described. Of the American texts consulted those which use either of these forms of the proposition are the following: Baker, Beman and Smith, Halsted, Hill, Holgate, Keigwin, McMahon, William B. Smith, The National Committee, Young and Schwartz, Dupuis.

Preceding the propositions related to this in American books is a treatment of measurement. A composite of the definitions of measurement, in fact usually the identical definition, is this: "To measure a quantity is to find how many times it contains another quantity of the same kind, called a unit of measure." One writer states it this way: "When we measure any magnitude the unit used must be of the same kind as the quantity to be measured." Please note the *must*. Then under angles he writes: "A center angle is measured by its intercepted arc." And again, ". . . but it does not mean that an angle is the same thing as an arc."

In an edition of Legendre translated by Brewster and revised by a West Point professor in 1828 we find that "a central angle is measured by its subtending arc." With this goes Legendre's apology: "It seems most natural to measure a quantity by a quantity of the same species. . . ."

In that elegantly written text by Chauvenet in 1869 we read: "The numerical measure of an angle at the center of a circle is the same as the numerical measure of its intercepted arc if the adopted unit of angle at the center is the angle at the center which intercepts the adopted unit of arc." Then follows Scholium I: "This theorem being of frequent application is usually more briefly, *though inaccurately,* expressed by saying that an angle at the center is measured by its intercepted arc. . . ." And Chauvenet uses the inaccurate wording.

One hyphenated text warns the student not to say "an angle at the center equals its intercepted arc because an angle and an arc are not quantities of the same kind." But the same text teaches the same student that "An angle at the center is measured by its intercepted arc."

Center angles are one case of angles formed by two secants of a circle. The "measured by" formula is passed on to the other angles of the group. And the merry game of defining measurement and immediately denaturing the definition proceeds indefinitely. What waste of time and numbing of the sensitiveness of students to the meanings of the words they use!

The meaning of *measure* in geometry is definite and will stand. Economy, consistency, demand that the definition be respected. Then "An inscribed angle equals one half of the center angle that intercepts the same arc," and "A center angle and its arc

have the same numeric." And similarly for the other angles of this group.

The following texts of recent issue have correct statements of this proposition differing from Euclid in wording: Stone-Millis has "A central angle has the same measure as its intercepted arc." Palmer-Taylor-Farnum has "A central angle equals in degrees its intercepted arc." Rolland R. Smith has "An inscribed angle is half as many degrees as its intercepted arc."

The British texts examined use invariably Euclid's proposition "A center angle is double of the inscribed angle on the same arc." One of the texts is a book written at the request of the British Association for the Improvement of Mathematical Teaching and would not be averse to adopting a novelty that would seem to be an improvement on Euclid. Two German texts—one of them a practical geometry—agree with the British practice.

Degrees appear to be regarded in the light of a fetish at present. Also the definition of degree and that of right angle are sometimes such that the pupil gains the idea that the relation of these two magnitudes is circular. "A right angle is an angle of ninety degrees" and "a degree is one ninetieth of a right angle" are definitions which allow the deduction that a degree is a degree and a right angle is a right angle.

A pupil can construct a right angle in a few seconds. He sees consciously or unconsciously thousands of right angles daily. "The right angle is by its nature the most simple unit of angle. Nevertheless custom has sanctioned a different unit" says Chauvenet. Therein he repeats Legendre. The definition of right angle and that of degree should be such as to leave no ambiguity in the pupil's mind. Since the right angle is a natural unit, and can be constructed so easily by the pupil while the degree is handed to him ready made, the right angle should be allowed the honor of more frequent appearance in geometric exercises.

"If two straight lines meet a third straight line so as to make the sum of the interior angles on the same side equal to two right angles, the two lines will be parallel." This is the statement of the proposition as given in Brewster's Legendre and in Davies' Legendre and is a faithful translation. Also, this is the way in which the proposition should be worded. Two lines taken at random and crossed by a third line are not known

to fill the required condition as to angles. The two lines desired parallel must be properly directed across a third. For this reason the statement should follow the order of construction as in Legendre.

Euclid begins the proposition "If a straight line falling on two straight lines makes . . ." and this form is used by almost all authors. The only exceptions in the books examined are Legendre, the Brewster and the Davies translations, one German text and one French, and their reading of the proposition must be preferred to that of the others.

A review by Professor Julian L. Coolidge of a report published by a subcommittee of the British Mathematical Association appeared in the *Mathematical Monthly* of June, 1924. The reviewer comments in the third person: "He has a wistful feeling that the committee had a splendid chance, which they failed to take, of recommending geometers to forget the cacophonous words *congruent* and *congruence.* Oh why did we ever allow these terms to escape from Pandora's box to trouble our lives!"

Farther on he quotes from a text of the 'Nineties: "In two congruent figures the parts of one figure equal respectively the parts of the other." The reviewer comments: "The distinction seems to be that the term 'congruent' applies to figures while the humbler and older term, 'equal,' applies to parts of figures."

One text of 1927 sides with the reviewer in using "equal" and "equivalent" in accordance with older usage. Two recent texts define "congruence" for closed figures, and "equal" for open figures, as sects and angles.

In the preface to a text of 1927 occurs this passage: "The author . . . has thought it wise to separate such axioms as 'Things equal to the same thing or to equal things are equal to each other' into two distinct axioms as follows: 'Things equal to equal things equal each other' and 'Things equal to the same thing equal each other.' This has a tendency to make the pupil more accurate in thought and expression."

This is a sensible emendation. Pupils can not be held accountable for unsportsmanly conduct if they shoot at the target both barrels of the weapon which is put into their hands.

"The greater side of every triangle subtends the greater angle" is a close translation of Euclid's proposition. Todhunter writes it "The greater side of every triangle has the greater

angle opposite to it.'' Loomis, Davies, Greenleaf, Bradbury, follow Euclid's wording. Auerbach-Walsh, Gore, Keigwin, Sykes-Comstock, Seymour, R. R. Smith, follow Euclid's form. Compare these with the common American form apparently beginning with Chauvenet. The National Committee has a good version: ''If two sides of a triangle are unequal, the greater side has the greater angle opposite it.'' The C. E. E. B. should have copied the Committee's form or Euclid's.

''Parallel with'' expresses the besideness of parallel lines as the expression commonly used does not. The preposition ''to'' in the common form is better seen in ''toward'' which reveals its fitness in ''perpendicular to.'' ''Parallel with'' comes into its own in the German ''parallel mit.''

''Letting fall'' perpendiculars or ''dropping'' them is quite uncommon to-day. Euclid's word means primarily to ''send down'' and is better entitled to that translation than to another. Todhunter shows good taste in translating ''draw a perpendicular.''

The proposition ''Parallel lines that intercept equal sects on one transversal intercept equal sects on any transversal'' has direct and simple proof in Chauvenet, Hawkes-Luby-Touton, Durell-Arnold, R. A. Avery, Betz-Webb, and R. R. Smith, by drawing the auxiliary parallels through the ends of the sects given equal. Some may prefer the trapezoid proof given in a few books.

One of the ideals in geometry is the minimum sequence of basic propositions. The minimum if attained would simplify the mechanics of geometry for both student and teacher. Many texts have what may be said to be minimal sequences. More transfer from line propositions to exercises is possible in any of these. Corollary propositions are subject to reduction just as much as any others. It is doubtful practice, the making corollaries of propositions that can not be placed in the base line.

Both the National Committee and the College Entrance Examination Board omit from their lists of approved propositions the following: ''Parallel sects intercepted by two parallel lines equal each other'' and ''Sects perpendicular to and intercepted by parallel lines equal each other.'' These usually occur as corollaries. The second of them is a special case of the first and is omitted from many books or carried as an exercise. The data

for either proposition give a parallelogram and the opposite sides of a parallelogram equal each the other. What need of another proposition?

Often grouped as a corollary with the two preceding is "A diagonal of a parallelogram forms with the sides equal triangles." It is used once after proof in finding the area of a triangle. But a proposition on equality of areas of two triangles could be used here just as well and a proposition saved thereby.

"Two angles whose sides are parallel respectively are equal or supplemental" is used once after proof in one well-organized book. But it was not needed in this place and so could be put in exercises. With it could go "Two angles whose sides are respectively perpendicular are equal or supplemental."

Most of the texts give as proposition or corollary "If the hypotenuse and an acute angle of one right triangle equal the like parts of another right triangle, the triangles equal each other." When this proposition comes after parallel lines its data make it very clearly a case of an earlier equality proposition. When it is used to introduce parallel lines it is easily replaced.

Writers are now unanimous for "If the sides of one triangle are proportional to the sides of another triangle, the triangles are similar." This is not used again and plainly belongs with the exercise propositions. There it was put by Bradbury in 1872. It was restored in 1892 without added usefulness.

The proposition on the sum of the angles of a triangle is loaded with nine corollaries in some cases, with eight in some cases, and so on down. Three corollaries is the maximum number needed. The others might be put among the exercises. The proposition on similarity of triangles which are mutually equiangular has as many as four corollaries attached none of which makes a live exercise.

In the text mentioned earlier is given "Two triangles whose sides are respectively parallel or respectively perpendicular are similar." This is not applied in the succeeding propositions. If met in exercises the figure can be interpreted easily to show the similarity.

The *got*-jargon invades the class in geometry as well as other classes. For example note well:

"I *got* to bisect angle *ACB*."

"How many sides has a dekagon *got?*"

"Line *AB* has *got* to cross side *CD*."

Readers of this paper are respectfully asked to take note of the extent to which this atrocious use of English enters into the speech of their students.

Rotation-translation as a proof of the sum of the angles of a triangle has some vogue, especially in junior high school texts. Record of it goes back as far as the beginning of the past century. Mach in *Space and Geometry* reports its use by Thibaut in Germany in 1809, maybe as originator. Four years later it appeared in the appendix to Playfair's Euclid, explained and commended. Playfair at some length explained the use of rotation in generating an angle but he was not so explicit regarding translation. Sir William Hamilton thought well of the proof. Casey in his edition of Euclid lauds Hamilton's "quaternion proof" and offers a simplified form of it.

In the United States Warren's *Primary Geometry* presents rotation-translation in 1887. Edwards in his text introduced the proof in the first few pages in 1895. Wentworth-Hill followed in 1901 in *First Steps in Geometry*. J. W. A. Young in 1906 in *Teaching of Mathematics* demonstrated the proof and recommended as follows: "This proof is easily understood, requires no explicit use of parallels, can come very early in the course, and at once opens the door for arithmetic and algebraic work."

The Breslich texts brought out at the University of Chicago use this proof, as do certain junior high school texts, the Schorling-Reeve *General Mathematics,* Stone's *The New Mathematics,* and Barber's *Junior High School Mathematics.*

Breslich in the set of books just published through the Macmillan Company seems to have discarded rotation-translation and uses a graphic proof of the traditional type.

Clark-Otis in their 1925 book offer rotation-translation to the secondary schools. This is given as the main proof and the standard proof is set as a "challenge" to the student. Both proofs follow parallel lines and either can be taken or both.

Does this proof prove? The names given show some able advocates and also some able opponents. In 1896 Halsted published his translation of Bolyai's *Science Absolute of Space.* In an appendix to this Halsted combated Playfair, Hamilton, Casey.

Edwards and all other advocates of rotation-translation. He quoted the argument of Perronet-Thompson against Playfair in 1830, Henrici against Hamilton in 1884, and Dodgson against the field in 1890. Casey in his Euclid in 1884 defends himself and his cobelievers against their critics. Henrici indicates the assumption which underlies the process, that *rotation is independent of translation*.

Probably most of us agree with Dodgson as quoted by Halsted that the process is "quite fascinating in its brevity and its elegance." Despite which Dodgson shows that he considers the process quite as illusive as his own creation, Alice in Wonderland, and vigorously exhibits the illusion.

It is noticeable that of all the books seen in this survey so few use rotation-translation. If secondary school books alone are considered, the percentage of users is very much lower than for junior high schools. Which raises the question, Is this method considered suitable for one group of schools and not for the other? Also, if suitable for the one group, why not suitable for the other?

Three somewhat differing forms of rotation-translation appear in the books. That given by Playfair, Casey, Edwards, is substantially as follows: Let ABC be a triangle and let BA, AC, CB, be extended to D, F, E, respectively. Let I and N be other names for A and B. Let IN rotate counterclockwise on pivot B until it lies in BE. Let IN move in EC until N is C and let it rotate on pivot C until it lies in CF. Then let IN move in FA until N is A and let it rotate on pivot A until it lies in AD. Now IN lies in the line and in the direction from which it started and therefore has rotated through four right angles. Also it has rotated through an exterior supplement of each interior angle. So the sum of the interior angles is two right angles. In this form N has been the center of rotation each time although it has taken the successive positions B, C, A.
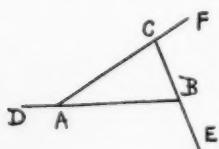
In the second form IN pivots on A and rotates counterclockwise until it lies in AC. Then IN moves in AC until N is C and it rotates until it lies in CB. Then IN moves in CB until I is B and rotates until it lies in BA. Now IN lies in the same line and in the opposite direction from which it started and has

rotated through two right angles. Also it has rotated through the sum of the interior angles, which equal therefore two right angles. The center of rotation has been I, N, and I, in turn.

The third form rotates IN on pivot A to AC, moves IN in AC until I is C, rotates on pivot C through the exterior angle vertical to the interior angle, moves IN on CB until I is B, and rotates it on pivot B until it lies on BA. Now IN lies in the same line and in the opposite direction from which it started. It has rotated through two interior angles and through the vertical of the other. Clark-Otis proves in this way.

The second form rotates directly through the sum of the interior angles but has the defect of shifting the center of rotation. This defect is remedied in the third form but a substitution is entailed. The first form keeps a constant center of rotation, rotates through one set of angles, but requires a subtraction in order to find the sum of the interior angles.

Very elementary indeed are the subjects treated in these notes. Needless is it to enlarge on our duty to the elementary because it is our duty to all built thereon. Accuracy in speech and simplicity in demonstration should be so taught that the student may have nothing to unlearn as he progresses. Nothing is so destructive of confidence, both subjective and objective, as unlearning. A great responsibility rests on authors to present faultless language and consistent logic because youth makes both the word and the thought of the book his own.

Be sure to fill in the blanks, cut out, and return the material called for on the last page of this number.

OPPORTUNITIES FOR TRANSFER MAKING IN ELEMENTARY MATHEMATICS

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I. Introduction

The title of this article is somewhat misleading. If the article has merit at all, the merit is not in the scientifically established validity of its position. If the reader expects to find here a study of the claims of the different theories concerning transfer of function, he will be disappointed, and had better turn at once to the next article. No references are made here to correlation coefficients, although instances of probable error are frequent.

The reader who has read this far deserves to be rewarded by references to two great mathematicians and teachers of mathematics: From Poincaré we quote:

"Is there anything that more needs fighting than dishonesty? —for this is one of the most common and degrading of vices of the natural man. Well! when we have caught the spirit of scientific method, of its scrupulous exactness, the horror of any suppression of fact, when we have grown to dread, as a stain upon our honour, the reproach of having even innocently cooked our results in the slightest degree—when this has become for us part of our professional pride, *shall we not carry into all our activities this desire for absolute sincerity, even to the point of no longer understanding what drives men to lie? And is not this the surest means to achieve the most rare and most difficult of all sincerities, that which consists in not deceiving one's self?"* (Italics are ours.)

We quote Bertrand Russell from memory to the effect that truth is only with the gods and that man must content himself with truthfulness.

Both Poincaré and Russell regard truthfulness as the cardinal virtue. Poincaré hopes that mathematics will serve as the discipline to bring about the desired outcome in man's mental discipline.

Much has been done in recent years toward anatomical studies of the elements in the curriculum. Skills have been listed, spe-

cific abilities have been charted, the sum of which is intended or tends, at least, to become the net result of the pupil's work in the given field. Without denying the character of the service of these efforts in the direction of explicitness and definiteness, we cannot but point out that they have their dangers. Anatomic study is perhaps too complimentary a term for some of these exercises. They tend to dissect the subject and to leave it dissected. The sum of the parts in this instance does not necessarily equal the whole, for the sum may be only a collection instead of an integration.

II. Transfer of Function

In the present armchair study of the problem we ask the reader to join us in leaving unsettled, as have indeed most laboratory psychologists left unsettled, the problem of the validity of the theory of transfer of function. We ask the reader also to make with us the following assumptions:

1. *Training tends to remain specific.* Even such traits as honesty or kindness are exercised by most individuals in specific situations only. Most men would not steal outright, yet a good recipe for obtaining an umbrella is to challenge the holder of an umbrella with the query—belligerently, of course—"Where did you get that umbrella?" It is not unlikely that the person thus challenged will give it up to you, convinced that it is your umbrella that he has "borrowed." Many a person, honest in most respects, is known to remember that the child who is really seven is only six years old, if six years is the age at which the railroads exempt children from paying fare.

2. *If transfer is to be accomplished, it must be practiced.* The story is commonplace, for it is typical of common experience: A child cannot find the name for a collection of two *apples* and three *apples*, because his teacher is in the habit of speaking of *oranges* in his exercises. The college student who differentiates Y with respect to X, in the mathematics class, finds it exasperatingly difficult to differentiate S with respect to T, in the physics class. If the transfer is to be made, the incidental aspect of a relationship should be varied, so as to emphasize the essential aspect. Only the genius *discovers* the general or universal aspect of a relationship. The great mass of road-takers must be given the practice in making the generalization or the transfer.

3. *There is, on the other hand, the danger that the novice will*

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make an undiscriminating transfer.   We report, by way of illustration, the case of a child who underwent a painful operation at the hands of a surgeon.   The experience brought on in him a condition that might come under the head of hysteria.   A few weeks later, when he was taken to a barber shop, he reacted to the white coat which the barber wore with the same hysterical manifestations.   ''White coat'' had become the symbol for ''pain-to-me.''

The assumptions in this discussion are therefore:

1. Training tends to be specific.

2. Transfers are made by pupils only if they are given practice in making transfers.

3. The undiscriminating person, meaning the modal person, tends at times to make the transfer to a situation the elements of which are similar, at best, only in form to those of the situation from which the generalization arose.

III.   Application of Assumptions

In the light of these assumptions we point out that, over and above the need for developing specific skills, we must show our students the broad implications of our science, and the need for being on guard against an undiscriminating generalization concerning the science and its possibilities.

By way of illustration we shall discuss the following specific aspects of mathematical method or device, and point out their applicability in other fields or the limitations of their applicability:

1. The method of exclusion, in which we have an instrument of doubtful extension value.

2. The use of epithets, in which we have an opportunity to confess to a fault.

3. The solution of equations, in which we can show that an efficient cause is not necessarily an exclusive cause.

4. The theory of function, in which we can emphasize the interrelatedness of each with all—in no purely figurative sense.

5   The postulates, in which we see mathematics as a normative science in the true sense, setting a standard for frankness in stating initial assumptions, instead (as is done in most other forms of discourse) of insinuating the assumptions.

1. *The method of exclusion* is used in plane geometry in a few

instances. Even in geometry it is well to reduce the number of instances to a minimum. In the case of the theorem, *If two angles of a triangle are unequal, the sides opposite them are unequal in the same order,* the method of exclusion is generally used, although the direct method is equally simple, and has an advantage over the indirect method in that it avoids for the learner the necessity of memorizing the place of this theorem, with reference to its converse.

But it is the opportunity that is offered to teachers when this theorem is being introduced that we wish to stress. This is an opportunity to discuss the reason for the applicability and the sufficiency of the method of exclusion in mathematics, and the inapplicability and the insufficiency of the method in other fields of discussion and discovery.

Why is the method applicable and sufficient in mathematics? Because the basis of comparison is clearly defined and the number of choices is limited. Two line segments or two angles or two arcs or two quantities of any given quality have but *one* of *three* possible relations: the first is greater than the second or it is equal to the second or it is less than the second. The elimination of any two limits us to the third.

How is the case in other fields of thought? It is like the one in mathematics only to the extent that the number of choices is limited and the basis of discrimination is defined. But these conditions are rarely met with in fields other than mathematics, and the fallacy occurs when so limited a method is applied in indiscriminate fashion in those other fields despite that fact. Hence the barrenness of discussions on such subjects as: *Who is the greater, Washington or Lincoln?* the basis of comparison not being defined, or *America or Guatemala or Country X is the greatest country in the world,* the basis of comparison not being defined and the number of possibilities being too unlimited to be considered economically in this fashion, or *The pen is mightier than the sword,* the terms again being very difficult to define, or *The mouth is for eating rather than speaking,* in which case both functions may characterize the organ rather than one function to the exclusion of the other, and where—not in the best circles, of course—both functions are sometimes manifested simultaneously.

It is our thesis that it is both an opportunity and a duty for the teacher of mathematics to "digress" from the immediate

theorem in hand in order to discuss the method of elimination or exclusion as a method, to point out that it has serious limitations and that it is fruitful of much fallacious argumentation where it is applied indiscriminately.

2. Mathematics has an opportunity to confess to faults—and does so confess in some of its nomenclature: the terms *imaginary, surd, transcendental, infinity* are indications all of the human origin of mathematics, for these terms betray the human tendency to use the epithet. An epithet either glorifies or belittles something that is not clearly understood. The terms which we have just listed are illustrations of this function of the epithet.

In the world of economics may be seen some of the worst aspects of the use of the epithet to describe observed phenomena. As it grows into a more nearly exact science it is not unlikely that economics will discard such capitalized names as Business Cycle, Iron Law of Wages, and the like.

The use of the epithet betrays an emotional state on the part of the observer. While observers cannot avoid the emotional quality in their experience, they should avoid or reduce to a minimum the vocabulary of emotion in describing correlations of phenomena.

Science used to be less discriminating when it enunciated that "Nature abhors a vacuum," an obviously meaningless collection of words which describes emotionally and explains not at all the phenomena which it was intended to correlate. Science used to talk about "chemical affinity."

Scientists are trying now to be more careful in this respect. In their reports they seek to avoid words that have emotional tinge. An instance of this scruple in the use of words to describe the behavior of particles is the introduction of the terms "tractate" and "pellate" as substitutes for the more vivid and, therefore, misleading words "attract" and "repel."

It is a serviceable digression to speak of these matters when the term *surd* or *imaginary number*, or any of the other terms to which we have referred, occurs in class for the first time. Mathematics, we are fond to say, is the science of necessary conclusions. Wherein is the necessity? The necessity must lie not in the satisfaction which is derived in venting one's feelings through the epithet. It lies rather in the compulsion under which we human beings find ourselves, when we think at all, to conform to

the laws of consistency. The limitations of this paper and (what is a more efficient cause) the limitations of the equipment of the present writer prevent him from considering at this point the manner in which the impulse to be consistent is in itself emotional in origin and character.

3. We come to a third opportunity for a worth-while digression. This arises when we introduce the pupil for the first time to the quadratic equation. Pupils often resist the notion that there are two or more roots or "correct" answers to a problem. "Which is correct?" they ask. The notion that either is correct or that both are correct is contrary to their usual and naïve experience. Here for the first time they meet an instance where an efficient cause is not necessarily an exclusive cause. A single effect may point to any one of a number of possible origins. Philosophically speaking, a single effect may have multiple causes, and a single cause may have multiple effects, any one of the members of these multiples being the only one observed for the time being. Is it far fetched to make the analogy to religious controversies, in which peoples have presumed to urge, as finalities, guesses as to causes of the phenomenon of existence itself, and to stamp, as heretical and perverse, guesses that are proposed by other peoples?

4. The fourth instance of an opportunity for serviceable digression of which we wish to write is the opportunity that arises in the discussion of the functional relation of two or more variables. We think of one sense datum as being a function of other sense data, when a change in the one datum is invariably accompanied by changes in the other data. $y = f(x)$ expresses in abstract form the mutual interpendence of each phenomenon with all others. This is a distinctive aspect of pantheism. It is what the poet means when he says:

> Flower in the crannied wall,
> I pluck you out of the crannies,
> I hold you here, root and all, in my hand,
> Little flower—but *if* I could understand
> What you are, root and all, and all in all,
> I should know what God and man is.

Events have remote and therefore unsuspected origins, far reaching and therefore unexpected effects. Splendid isolation of self from other selves, of a people from other peoples, may be a ques-

tionable or an unquestionable ideal.   Unquestionably, complete
isolation is a practical impossibility.   When we realize that our
lives are penetrated by and penetrate the lives of others in an in-
extricable cause and effect relationship, we are likely to see in our
every social relation opportunity and responsibility that spell
human dignity.

5. The fifth and last of the list of purposeful digressions which
we have selected arbitrarily is the consideration of the postulate
as a mathematical prejudice.   To the reader who has a prejudice
against prejudices, this classification is something of a shock.
"One geometry," says Poincaré, "cannot be more true than an-
other; it can only be *more convenient.*"   Such is the case with
prejudices or postulates.   Their fruit in terms of consistent sys-
tems and correspondence with measurable experience will deter-
mine the degree to which they are convenient.   The notion of
exclusive effect or cause, the use of the epithet whereby the
strange is either anathematized or canonized, are examples of
prejudices that we usually do not explicitly label.

Most or nearly all of our prejudices are hidden, sometimes even
from ourselves.   Bacon has attempted to classify them as the
idols, the idol of the cave, the idol of the tribe, the idol of the
theater, the idol of the market place.   Our personal adjustments
and other values are invested in these prejudices.   Sometimes
without knowing it, we resent the attempts of adversaries in a
discussion to uncover them.   We then indulge in indignant and
high-pitched assertions to the following effect: "Every sensible
person knows that," or "*Naturally* such and such is the case,"
or "It is self-evident," or "Any fool knows that," or "It goes
without saying," and so forth, and so on.   These ejaculations
are often panicky attempts to hide from our hearers and from
ourselves, as well, the fact that our position is logically not
tenable or that, at any rate, we have never before examined the
logical implications of it.

We shall permit ourselves to relate two stories which are in-
tended to illustrate the manner in which we carry about with us
unwittingly prejudices which we repudiate as soon as we become
aware of them.

The first of these is of a lady who went shopping in a depart-
ment store of a large city.   She had her little child with her and
took advantage of the kindergarten which the department store
places at the disposal of its patrons, leaving the child there

among other children in order that she might the more expeditiously accomplish her shopping mission. The lady took longer than she expected in making the rounds of the departments. She heard the bell sound the signal for closing, and hastened to the kindergarten for her child. There was but one child left, and that, a colored child. At this point the reader is likely to indulge in sympathy for the lady. But she needs no sympathy. The child is hers. The lady is colored, too.

The second story is less dramatic but equally illustrative: A beggar has a brother, but the brother has no brother. The answer, of course, is that the brother has a sister. Our beggar is a woman.

The point of these stories, like that of so many that take us unawares, is that unconsciously we use synonymously terms that are repositories of prejudices. A lady, for the average white Nordic, is not reasonably to be thought other than white. A beggar, in our ordinary experience, is usually a man.

Many a debate, in high schools and in congress, would lose its raison d'être were it not for the equivocal use of terms, the equivocation having its origin in prejudgments that none of the participants bothers to make explicit. Witness the following:

Editorial in the New York Sun, October 14, 1924

Professor IRVING FISHER of Yale has come out for JOHN W. DAVIS

Some of the SUN's readers may have forgotten who Professor IRVING FISHER is. Let us freshen their memories.

A month after the death of the late President HARDING, Professor IRVING FISHER, who was stumping the country for the League of Nations Non-Partisan Association, delivered a speech in East Liverpool, Ohio, in which he declared that during the campaign of 1920 Senator HARDING said to him, first pledging him to secrecy until after election: "I want the United States to get into the League of Nations as much as you do."

This was equivalent to accusing the dead President of having sailed under false colors in the campaign of 1920. Nobody believed FISHER and everybody condemned him for uttering this shocking statement about a man who could no longer answer.

So far as the SUN knows, Professor IRVING FISHER has never retracted this slanderous statement.

Now FISHER announces that he is supporting JOHN W. DAVIS and one of the reasons he gives for his stand is that DAVIS is "one who can restore honesty."

Can Mr. DAVIS restore the honesty which Professor FISHER lost a year ago?

New York World, January 19, 1925

Pointing out that the amendment (the proposed Child Labor Amendment) would give lawmakers authority to prohibit the labor of all persons under eighteen, Mr. Marshall said:

"I worked before I was eighteen years old, and my only regret is that I did not do more manual labor when I was young. You would not have so many gunmen if the boys of to-day were taught to work. The old-fashioned virtue of industry is what we lack in this age."

. . . .

"Remember that the State of New York is not beloved in all parts of the South and West. If the proponents of this measure are sufficiently successful, no one can tell what some might try to do to destroy our prosperity by passing similar laws hampering our activities in the State."

From School and Society, March 6, 1926

Using the slogan "Wanted: Brains at the Top," the associate and district superintendents and high school principals of New York City have issued a pamphlet setting forth their arguments for increases in salaries approximately $5,000.00 higher in each case than those allotted them in the Ricca teachers' salary bill which was introduced in the State Legislature last week. . . .

The pamphlet points out that the salaries sought are only a small percentage of the expenditures which the supervisors control. It is stated for example that the superintendent of schools is at the head of an organization spending $105,000,000.00 a year, and that if he drops in efficiency but 1 per cent., the loss to the taxpayers is $1,000,000.00. It continues this line of argument for the positions of associate superintendent, district superintendent and high school principal. . . .

These clippings can, no doubt, be matched by hosts of others. In the first, notice the implication that a man who is dead could not have been guilty of having made a secret pledge, the gradual insinuation that any person who makes an assertion to the contrary is necessarily dishonest, and finally, that if such a person supports a candidate for office, the candidate himself becomes disqualified.

The other articles quoted are worthwhile exercises in the discovery of unexpressed but vital assumptions.

Herein lies the distinctive character of the postulates of the mathematics. They are prejudgments, convenient prejudgments, made explicit through careful tabulation, in advance of their use

in the elaboration that constitutes much of our mathematical thinking.

The student who will not accept the postulates of mathematics is within his rights in all logic to reject them. He simply deprives himself of the fruits of these assumptions. But the student who accepts them is in a position to examine their elaborations into a system which up to the present has, on the whole, squared with experience.

Parenthetically, we venture to urge that we teachers think of postulates not as self-evident truths (the phrase "self-evident truth" is really a logical and psychological self-contradiction). Rather should we speak of the postulates as statements which we accept without proof.

Mathematics is *the* normative science, because of the autonomous character of its development, as contrasted with the empirical reasoning that necessarily characterizes the natural sciences and the instinctive reasoning that characterizes our everyday adjustments.

Professor Keyser, in his *Thinking about Thinking*, elaborates this idea in interesting fashion. Psychologists and philosophers might take issue with him in his divisions of types of reasoning. We must agree with him however that some types of reasoning are predominantly autonomous, where the reasoning is from accepted premises and definitions, some types are predominantly empirical, where the reasoning is almost entirely from observation that is determined by the character of the instrument and of the observer, and finally some types are predominantly instinctive, where the stimulus-response relation occurs with little or no gap for conscious and deliberate stating of conclusion on the basis of listed elements in the stimulus-situation.

Every science which seeks to examine the grounds of its own development—and every science at some time in its growth pauses to do so, if only to take stock and to decide the next step—approaches the mathematical science in form. Physics and chemistry thus tend to become mathematical.

IV. Summary and Conclusions

We began with a statement of assumptions that training tends to be specific, that discriminating transfer making must be practised, and that transfer making is often practised indiscriminately.

We have considered five of the opportunities which we find in mathematics teaching for the making of such transfers or digressions, to wit:

1. The method of exclusion, which has its limitations and uses in other fields,
2. the epithet, which is found even in mathematics,
3. the quadratic equation, which gives intimations that a sufficient cause is not necessarily the cause,
4. the theory of function, which gives intimations of the interconnectedness of each with all, physically and socially, and
5. the postulates, which are examples of what we should try to do with our hidden prejudices in order to achieve communion with and understanding of our fellows.

Discussions in class of the topics which we have thus outlined are admittedly serious digressions. Digressions from what? It depends on our conception of the purpose and the function of mathematics teaching. We must develop skills in mathematical manipulations. The time which is at our disposal is limited. In view of the limited time at our disposal, is it worth trying to introduce in elementary classes the broader implications of our subject? Can they be introduced with any hope of achieving any significant results?

In his *Introduction to Mathematics,* Professor Whitehead tells us that much of the difficulty of mathematics teaching can be attributed to its extreme preoccupation with the trivialities of problem and equation solving. Would it not be worth our while—worth our while in the best sense—to displace some of these trivialities with such types of exercises as we have indicated? Recent changes in the content of courses seem to give hints of a movement in that direction.

In a later article we shall analyze these changes from this point of view.

NEWS AND NOTES

The Louisiana-Mississippi Section of the Mathematical Association of America and the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics held a joint meeting at Jackson, Mississippi, on March 30 and 31.

The following program was given:

At Central High School

W. C. Roaten, presiding. Friday: 3:30 P.M., Address of Welcome, Superintendent E. L. Bailey, Jackson, Miss.; Response, W. C. Roaten, DeRidder, Louisiana. A Play: "Geometry Humanized," written for MATHEMATICS TEACHER by Erma Scott, Greeley, Colorado, acted by students of Central High, Jackson, Mississippi. "Creating an Interest in Plane Geometry," Mrs. B. A. Summer, Columbia, Mississippi. Discussion, "The Types of Written Examination in Plane Geometry and Their Relative Values," Miss Norma Touchstone, Alexandria, La. Discussion, "College Entrance Requirements in Mathematics," C. D. Smith, Louisiana College, La., W. C. Roaten, High School, DeRidder, La. 6:00 P.M., Buses and automobiles convey visitors to Mississippi College, Clinton, eight miles from Jackson. Points of departure: Central High School and Edwards Hotel.

At Mississippi College

President J. W. Provine, presiding. 7:00 P.M., Dinner. Welcome Address, President J. W. Provine. From the Administrative Viewpoint, President D. M. Key, Millsaps College, Jackson, Miss.; President C. A. Gillespie, Belhaven College, Jackson, Missisipi; President B. M. Walker, A. & M. College, Mississippi; Dean W. H. Zeigel, Miss. Delta State Teachers College, Cleveland, Mississippi. Response for Section and Council, S. T. Sanders, Chairman. 8:30 P.M., "Mathematics and Civilization," Chas. N. Moore, Professor of Mathematics, University of Cincinnati, and Official Delegate from the Mathematical Association of America.

At Belhaven College

S. T. Sanders, presiding. Saturday: 9:00 A.M., "A Quadratic One-to-one Transformation between Two Planes," President B. M. Walker, A. & M. College, Mississippi; "The Construction of Certain Magic Squares," J. R. Hitt, Professor of Mathematics, Mississippi College, Miss.; "A Problem in Geometry," Jas. P. Cole, Professor of Mathematics, L. P. I., Louisiana; "Some Coincidences in Mathematics," Chas. N. Wunder, Professor of Mathematics, University of Mississippi; "The Chords of Ptolemy," H. E. Buchanan, Professor of Mathematics, Tulane University; "The Mathematics News Letter," S. T. Sanders, Professor of

Mathematics, Louisiana State University. 11:30 A.M., Business Session of the Section, Business Session of the Council. 1:00 P M., Social Hour at Millsaps College (Dean B. E. Mitchell, in charge).

A MESSAGE FROM THE NEW PRESIDENT

There are three country-wide organizations of teachers of mathematics in the United States. In the field of research, The American Mathematical Society (R. G. D. Richardson, Secretary, 501 W. 116th St., New York City). In the field of college teaching, The Mathematical Association of America (W. D. Cairns, Secretary, Oberlin, Ohio). In the field of elementary and secondary school mathematics, The National Council of Teachers of Mathematics (J. A. Foberg, Secretary, California, Penna.).

Each of these organizations publishes one or more journals, and together they cover the field from grade seven through the most advanced research carried on in this country. Every teacher of mathematics is urged to belong to one or more of them for his own good and for the good of the mathematics cause.

The membership dues of the National Council of Teachers of Mathematics are $2.00 a year, for which the MATHEMATICS TEACHER (8 issues a year) is mailed to every member.

The National Council of Teachers of Mathematics is in effect a federation of local clubs and associations of teachers of mathematics all over this country. All who are interested are urged to join local clubs if possible; to organize local clubs or branches where none exist; to join the National Council of Teachers of Mathematics either directly or through their local clubs; and to encourage others to join.

Apply for membership to the MATHEMATICS TEACHER, 525 West 120th Street, New York City.

Members are asked to extend this invitation, and to supply the address of the MATHEMATICS TEACHER to all who may be interested. The editors have set a goal of 10,000 members for the next two years. They give their time without stint, and deserve our cordial support.

HARRY C. BARBER